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Relations for virtual fundamental classes of Hilbert schemes of curves on surfaces

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Abstract. In [2] we constructed virtual fundamental classes $[[\mathrm{Hilb}_V^m]]$ for Hilbert schemes of divisors of topological type m on a surface V, and used these classes to define the Poincaré invariant of V:

$$(P_V^+, P_V^-) : H^2(V, \mathbb{Z}) \longrightarrow \Lambda^* H^1(V, \mathbb{Z}) \times \Lambda^* H^1(V, \mathbb{Z}).$$

We conjecture that this invariant coincides with the full Seiberg–Witten invariant computed with respect to the canonical orientation data.

In this note we prove that the existence of an integral curve $C \subset V$ induces relations between some of these virtual fundamental classes $[[\mathrm{Hilb}_V^m]]$. The corresponding relations for the Poincaré invariant can be considered as algebraic analoga of the fundamental relations obtained in [6].

1 Introduction

The symplectic Thom conjecture for homology classes with negative self-intersection, proven by Ozsváth and Szabó, is an immediate consequence of the following two facts:

i) Taubes' constraints for the Seiberg–Witten basic classes of a closed symplectic four-manifold [7].
ii) A fundamental relation between certain Seiberg–Witten invariants, which arises from embedded surfaces with negative self-intersection, due to Ozsváth and Szabó [6].

In this note we prove an analoguous relation for the virtual fundamental classes of certain Hilbert schemes of algebraic curves on smooth projective surfaces. To be more precise: Let V be a smooth connected projective surface over $\mathbb{C}$ and let $k := c_1(\mathcal{K}_V) \in H^2(V, \mathbb{Z})$ be the first Chern class of its canonical line bundle. For any class $m \in H^2(V, \mathbb{Z})$ we have the Hilbert scheme Hilb_V^m parametrizing effective divisors $D \subset V$ with $c_1(\mathcal{O}_V(D)) = m$. In [2] we constructed a virtual fundamental class $[[\mathrm{Hilb}_V^m]] \in A_{\frac{1}{2}m(m-k)}(\mathrm{Hilb}_V^m)$ in the

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Chow group of Hilb_V^m. By abuse of notation we will denote the image of $[[\mathrm{Hilb}_V^m]]$ under the cycle map

$$A_{\frac{1}{2}m(m-k)}(\mathrm{Hilb}_V^m) \longrightarrow H_{m(m-k)}(\mathrm{Hilb}_V^m, \mathbb{Z})$$

by the same symbol. Note that there exists a natural morphism $\rho : \mathrm{Hilb}_V^m \to \mathrm{Pic}_V^m$ sending a divisor $D \subset V$ to the class $[\mathcal{O}_V(D)]$ of its associated line bundle. Let $\mathbb{D} \subset \mathrm{Hilb}_V^m \times V$ be the universal divisor, and put $u := c_1(\mathcal{O}_V(\mathbb{D})|_{\mathrm{Hilb}_V^m \times \{p\}})$, where $p \in V$ is an arbitrary point.

Consider now a reduced and irreducible curve $C \subset V$, set $c := c_1(\mathcal{O}_V(C))$, and denote by $\kappa_c \in \Lambda^2 H^1(V, \mathbb{Z})^\vee$ the map

$$\begin{aligned} \kappa_c : \Lambda^2 H^1(V, \mathbb{Z}) &\longrightarrow \mathbb{Z} \\ a \wedge b &\longmapsto \langle a \cup b \cup c, [V] \rangle. \end{aligned}$$

Note that we have a canonical isomorphism

$$\Lambda^* H^1(V, \mathbb{Z})^\vee \xrightarrow{\cong} H^*(\mathrm{Pic}_V^m, \mathbb{Z}),$$

which we use to identify both groups. Hence we get a class $\kappa_c \in H^2(\mathrm{Pic}_V^m, \mathbb{Z})$.

Let $\iota : \mathrm{Hilb}_V^{m-c} \to \mathrm{Hilb}_V^m$ be the closed embedding sending $D' \in \mathrm{Hilb}_V^{m-c}$ to $D' + C \in \mathrm{Hilb}_V^m$. Our main result relates the homology classes $[[\mathrm{Hilb}_V^m]]$ and $[[\mathrm{Hilb}_V^{m-c}]]$ when $m \cdot c < 0$, and $[[\mathrm{Hilb}_V^m]]$ and $[[\mathrm{Hilb}_V^{m+c}]]$ when $(k - m) \cdot c < 0$.

Theorem 1. *Let V be a surface, and fix a class $m \in H^2(V, \mathbb{Z})$. Let $C \subset V$ be a reduced and irreducible curve, and set $c := c_1(\mathcal{O}_V(C))$.*

i) *Suppose that $m \cdot c < 0$, and denote by ρ the map $\mathrm{Hilb}_V^m \to \mathrm{Pic}_V^m$. Let $\iota : \mathrm{Hilb}_V^{m-c} \to \mathrm{Hilb}_V^m$ be the inclusion given by the addition $D \mapsto D + C$. Then we have*

$$[[\mathrm{Hilb}_V^m]] = \left(\sum_i \rho^* \left(\frac{\kappa_c^i}{i!} \right) \cdot u^{\frac{1}{2}(c^2 + c \cdot m) - m \cdot c - i} \right) \cap \iota_* [[\mathrm{Hilb}^{m-c}]].$$

ii) *Suppose that $(k - m) \cdot c < 0$, and denote by $\tilde{\rho}$ the map $\mathrm{Hilb}_V^{m+c} \to \mathrm{Pic}_V^{m+c}$. Let $\iota : \mathrm{Hilb}_V^m \to \mathrm{Hilb}_V^{m+c}$ be the inclusion given by the addition $D \mapsto D + C$. Then we have*

$$\iota_* [[\mathrm{Hilb}_V^m]] = \left(\sum_i \tilde{\rho}^* \left(\frac{(-\kappa_c)^i}{i!} \right) \cdot u^{\frac{1}{2}(c^2 + c \cdot k) - (k-m)c - i} \right) \cap [[\mathrm{Hilb}_V^{m+c}]].$$

In [2] we used the virtual fundamental classes $[[\mathrm{Hilb}_V^m]]$ to define a map

$$(P_V^+, P_V^-) : H^2(V, \mathbb{Z}) \longrightarrow \Lambda^* H^1(V, \mathbb{Z}) \times \Lambda^* H^1(V, \mathbb{Z})$$

which we called the Poincaré invariant of V. This map is invariant under smooth deformations of V, satisfies a blow-up formula, and a wall crossing formula for surfaces with $p_g(V) = 0$. We conjecture that the Poincaré invariant coincides with the full Seiberg–Witten invariant of [5] computed with respect to the canonical orientation data. Our relations between the virtual fundamental classes of Hilbert schemes lead to corresponding relations for the Poincaré invariant. In order to express these relations neatly, we introduce truncation maps τ_n: for an integer n we define

$$\tau_n : \Lambda^* H^1(V,\mathbb{Z}) \longrightarrow \Lambda^* H^1(V,\mathbb{Z})$$

as follows: when $P = \sum_i P_i$ is the decomposition of a form P into its homogeneous components $P_i \in \Lambda^i H^1(V,\mathbb{Z})$, then

$$\tau_n(P) := \sum_{i=0}^{n} P_i.$$

Corresponding to the identification $H^*(\mathrm{Pic}^m_V,\mathbb{Z}) = \Lambda^* H^1(V,\mathbb{Z})^\vee$ we have an identification $H_*(\mathrm{Pic}^m_V,\mathbb{Z}) = \Lambda^* H^1(V,\mathbb{Z})$.

With these notations the relations for the Poincaré invariant read as follows:

Theorem 2. *Let V be a surface, and fix a class $m \in H^2(V,\mathbb{Z})$. Let $C \subset V$ be a reduced and irreducible curve, and set $c := c_1(\mathcal{O}_V(C))$.*

i) *If $m \cdot c < 0$, then*

$$P^{\pm}_V(m) = \tau_{m(m-k)}\big(\exp(\kappa_c) \cap P^{\pm}_V(m-c)\big).$$

ii) *If $(k-m)\cdot c < 0$, then*

$$P^{\pm}_V(m) = \tau_{m(m-k)}\big(\exp(-\kappa_c) \cap P^{\pm}_V(m+c)\big).$$

This result can be considered as an algebraic analog of the Ozsváth–Szabó relation, as we will explain in Section 4 below.

2 Comparing virtual fundamental classes of Hilbert schemes

In this paper all surfaces will be smooth, projective, connected, and defined over the field of complex numbers. We denote by $k := c_1(\mathcal{K}_V)$ the first Chern class of the canonical line bundle of a surface V.

Recall that an element $c \in H^2(V,\mathbb{Z})$ is *characteristic* iff $c \equiv k \mod 2$. For a characteristic element $c \in H^2(V,\mathbb{Z})$, we denote by $\theta_c \in \Lambda^2 H^1(V,\mathbb{Z})^\vee$ the map

$$\begin{aligned} \theta_c : \Lambda^2 H^1(V,\mathbb{Z}) &\longrightarrow \mathbb{Z} \\ a \wedge b &\longmapsto \frac{1}{2}\langle a \cup b \cup c, [V]\rangle. \end{aligned}$$

We define $\xi_V \in \Lambda^4 H^1(V,\mathbb{Z})^\vee$ to be the map

$$\begin{aligned} \xi_V : \Lambda^4 H^1(V,\mathbb{Z}) &\longrightarrow \mathbb{Z} \\ a \wedge b \wedge c \wedge d &\longmapsto \langle a \cup b \cup c \cup d, [V]\rangle. \end{aligned}$$

Using the identification

$$\Lambda^* H^1(V,\mathbb{Z})^\vee = H^*(\mathrm{Pic}^m_V,\mathbb{Z}),$$

we obtain classes $\theta_c \in H^2(\mathrm{Pic}^m_V,\mathbb{Z})$ and $\xi_V \in H^4(\mathrm{Pic}^m_V,\mathbb{Z})$.

Lemma 3. *Let V be a surface, and fix a class $m \in \mathrm{NS}(V)$ in the Neron Severi group. Choose a normalized Poincaré line bundle $\mathbb{L}$ on $\mathrm{Pic}_V^m \times V$, and let $\mu : \mathrm{Pic}_V^m \times V \to \mathrm{Pic}_V^m$ be the projection. Then we have*

$$ch(\mu_!\mathbb{L}) = \chi(\mathcal{O}_V) + \frac{1}{2}m(m-k) - \theta_{2m-k} + \xi_V.$$

Proof. By the Grothendieck–Riemann–Roch theorem [3, Theorem 15.2] we have

$$td(\mathrm{Pic}_V^m) \cdot ch(\mu_!\mathbb{L}) = \mu_!\big\{td(\mathrm{Pic}_V^m \times V) \cdot ch(\mathbb{L})\big\}.$$

Hence we need to compute those components of the expression $td(\mathrm{Pic}_V^m \times V) \cdot ch(\mathbb{L})$ which have bidegree $(*, 4)$ with respect to the decomposition

$$\begin{aligned} H^*(\mathrm{Pic}_V^m \times V, \mathbb{Z}) &\cong H^*(\mathrm{Pic}_V^m, \mathbb{Z}) \otimes H^*(V, \mathbb{Z}) \\ &\cong \Lambda^* H^1(V, \mathbb{Z})^\vee \otimes H^*(V, \mathbb{Z}). \end{aligned}$$

Set $f := c_1(\mathbb{L})$. Then

$$\begin{aligned} f^{2,0} &= 0 \in H^2(\mathrm{Pic}_V^m, \mathbb{Z}), \\ f^{1,1} &= \mathrm{id} \in \mathrm{Hom}(H^1(V, \mathbb{Z}), H^1(V, \mathbb{Z})), \\ f^{0,2} &= m \in H^2(V, \mathbb{Z}), \end{aligned}$$

where the first equality holds since $\mathbb{L}$ is normalized.

Next we compute $g := f^2$. We obtain

$$\begin{aligned} g^{2,2} &= -2 \cdot (a \wedge b \mapsto a \cup b) \in \mathrm{Hom}(\Lambda^2 H^1(V, \mathbb{Z}), H^2(V, \mathbb{Z})), \\ g^{1,3} &= 2 \cdot (a \mapsto a \cup m) \in \mathrm{Hom}(H^1(V, \mathbb{Z}), H^3(V, \mathbb{Z})), \\ g^{0,4} &= m \cup m \in H^4(V, \mathbb{Z}), \end{aligned}$$

all other components being zero. Here the first equality needs justification. Choose a basis $v_1, \dots, v_{2q}$ of $H^1(V, \mathbb{Z})$, and denote by $w_1, \dots, w_{2q}$ the dual basis of $H^1(V, \mathbb{Z})^\vee$. Then

$$f^{1,1} = \sum_i w_i \otimes v_i,$$

and

$$\begin{aligned} g^{2,2} = \left(f^{1,1}\right)^2 &= \Big(\sum_i w_i \otimes v_i\Big) \cup \Big(\sum_i w_i \otimes v_i\Big) \\ &= -\sum_i \sum_j (w_i \wedge w_j) \otimes (v_i \cup v_j) = -2 \sum_{i<j} (w_i \wedge w_j) \otimes (v_i \cup v_j). \end{aligned}$$

Now we compute the component of f^3 of bidegree $(2, 4)$, the only component that does not vanish. We find

$$\begin{aligned} f^3 &= 3(f^{1,1})^2 \cup f^{0,2} \\ &= -6 \cdot (a \wedge b \mapsto a \cup b \cup m) \in \mathrm{Hom}(\Lambda^2 H^1(V, \mathbb{Z}), H^4(V, \mathbb{Z})). \end{aligned}$$

Finally we obtain

$$
\begin{aligned}
f^4 = (f^{1,1})^4 &= \sum_{i,j,k,l} (w_i \wedge w_j \wedge w_k \wedge w_l) \otimes (v_i \cup v_j \cup v_k \cup v_l) \\
&= 24 \left(\sum_{i<j<k<l} (w_i \wedge w_j \wedge w_k \wedge w_l) \otimes (v_i \cup v_j \cup v_k \cup v_l) \right) \\
&= 24(a \wedge b \wedge c \wedge d \mapsto a \cup b \cup c \cup d).
\end{aligned}
$$

Since $td(\mathrm{Pic}_V^m) = 1$, we get

$$
td(\mathrm{Pic}_V^m \times V) = pr_V^* td(V) = pr_V^*(1 - \frac{1}{2}k + \chi(\mathcal{O}_V) \cdot \mathrm{PD}[pt]),
$$

where $pr_V : \mathrm{Pic}_V^m \times V \to V$ denotes the projection onto V.

Putting everything together, we get

$$
\begin{aligned}
ch(\mu_! \mathbb{L}) &= \left\{ \exp f \cup pr_V^* \left(1 - \frac{k}{2} + \chi(\mathcal{O}_V) \cdot \mathrm{PD}[pt] \right) \right\} / [V] \\
&= \left\{ (\exp f)^{*,4} - (\exp f)^{*,2} \cup pr_V^* \frac{k}{2} + \chi(\mathcal{O}_V) \cdot \mathrm{PD}[pt] \right\} / [V] \\
&= \chi(\mathcal{O}_V) + \frac{1}{2} m \cdot (m-k) - \theta_{2m-k} + \xi_V. \qquad \square
\end{aligned}
$$

For an arbitrary element $c \in H^2(V, \mathbb{Z})$, we denote by $\kappa_c \in \Lambda^2 H^1(V, \mathbb{Z})^\vee$ the map

$$
\begin{aligned}
\kappa_c : \Lambda^2 H^1(V, \mathbb{Z}) &\longrightarrow \mathbb{Z} \\
a \wedge b &\longmapsto \langle a \cup b \cup c, [V] \rangle,
\end{aligned}
$$

and also the corresponding class in $H^2(\mathrm{Pic}_V^m, \mathbb{Z})$.

Corollary 4. *Let V be a surface, and fix two classes $m, c \in \mathrm{NS}(V)$. Choose a normalized Poincaré line bundle $\mathbb{L}$ on $\mathrm{Pic}_V^m \times V$ and a line bundle $\mathcal{L}_c$ on V with $c_1(\mathcal{L}_c) = c$. Let $\mu : \mathrm{Pic}_V^m \times V \to \mathrm{Pic}_V^m$ and $pr_V : \mathrm{Pic}_V^m \times V \to V$ be the projections. Then*

$$
\begin{aligned}
ch(\mu_! \mathbb{L} - \mu_!(\mathbb{L} \otimes pr_V^* \mathcal{L}_c^\vee)) &= m \cdot c - \frac{1}{2}(c^2 + c \cdot k) - \kappa_c, \\
c(\mu_! \mathbb{L} - \mu_!(\mathbb{L} \otimes pr_V^* \mathcal{L}_c^\vee)) &= \exp(-\kappa_c).
\end{aligned}
$$

Proof. The assertion concerning the Chern character is a direct consequence of Lemma 3. The formula for the Chern class follows immediately since $H^*(\mathrm{Pic}_V^m, \mathbb{Z})$ has no torsion. $\square$

In order to state our main result, we have to recall some facts from [2]. For a surface V and a class $m \in H^2(V, \mathbb{Z})$, we denote by Hilb_V^m the Hilbert scheme of divisors D with $c_1(\mathcal{O}_V(D)) = m$. Let $\mathbb{D} \subset \mathrm{Hilb}_V^m \times V$ be the universal divisor, and denote by

$\pi : \mathrm{Hilb}_V^m \times V \to \mathrm{Hilb}_V^m$ the projection onto Hilb_V^m. In [2], we constructed an obstruction theory (in the sense of Behrend and Fantechi [1])

$$\varphi : R^\bullet \pi_* \mathcal{O}_{\mathbb{D}}(\mathbb{D})^\vee \to \mathcal{L}^\bullet_{\mathrm{Hilb}_V^m}$$

for Hilb_V^m, and showed that this obstruction theory defines a virtual fundamental class

$$[[\mathrm{Hilb}_V^m]] \in A_{\frac{1}{2}m(m-k)}(\mathrm{Hilb}_V^m).$$

We refer to [2] for the construction and properties of this class. Choose a point $p \in V$ and set

$$u := c_1(\mathcal{O}(\mathbb{D})|_{\mathrm{Hilb}_V^m \times \{p\}}).$$

Recall that we use the same symbol $[[\mathrm{Hilb}_V^m]]$ for the image of the virtual fundamental class under the cycle map.

Theorem 5. *Let V be a surface, and fix a class $m \in H^2(V, \mathbb{Z})$. Let $C \subset V$ be a reduced and irreducible curve, and set $c := c_1(\mathcal{O}_V(C))$.*

i) *Suppose that $m \cdot c < 0$, and denote by ρ the map $\mathrm{Hilb}_V^m \to \mathrm{Pic}_V^m$. Let $\iota : \mathrm{Hilb}_V^{m-c} \to \mathrm{Hilb}_V^m$ be the inclusion given by the addition $D \mapsto D + C$. Then we have*

$$[[\mathrm{Hilb}_V^m]] = \left(\sum_i \rho^* \left(\frac{\kappa_c^i}{i!} \right) \cdot u^{\frac{1}{2}(c^2 + c \cdot m) - m \cdot c - i} \right) \cap \iota_*[[\mathrm{Hilb}^{m-c}]].$$

ii) *Suppose that $(k - m) \cdot c < 0$, and denote by $\tilde{\rho}$ the map $\mathrm{Hilb}_V^{m+c} \to \mathrm{Pic}_V^{m+c}$. Let $\iota : \mathrm{Hilb}_V^m \to \mathrm{Hilb}_V^{m+c}$ be the inclusion given by the addition $D \mapsto D + C$. Then we have*

$$\iota_*[[\mathrm{Hilb}_V^m]] = \left(\sum_i \tilde{\rho}^* \left(\frac{(-\kappa_c)^i}{i!} \right) \cdot u^{\frac{1}{2}(c^2 + c \cdot k) - (k-m)c - i} \right) \cap [[\mathrm{Hilb}_V^{m+c}]].$$

Proof. Suppose first that $m \cdot c < 0$. Then we have $H^0(\mathcal{O}_C(D)) = 0$ for any divisor $D \in \mathrm{Hilb}_V^m$. It follows that the inclusion $\mathrm{Hilb}_V^{m-c} \to \mathrm{Hilb}_V^m$ is an isomorphism. However, the obstruction theories differ: Denote by $\mathbb{C}$ the product $\mathrm{Hilb}_V^m \times C$. The short exact sequence

$$0 \to \mathcal{O}_{\mathbb{D}-\mathbb{C}}(\mathbb{D} - \mathbb{C}) \to \mathcal{O}_{\mathbb{D}}(\mathbb{D}) \to \mathcal{O}_{\mathbb{C}}(\mathbb{D}) \to 0$$

gives rise to a distinguished triangle:

$$\begin{tikzcd}
R^\bullet \pi_* \mathcal{O}_{\mathbb{D}-\mathbb{C}}(\mathbb{D} - \mathbb{C}) \arrow[r] & R^\bullet \pi_* \mathcal{O}_{\mathbb{D}}(\mathbb{D}) \arrow[d] \\
 & R^\bullet \pi_* \mathcal{O}_{\mathbb{C}}(\mathbb{D}) \arrow[ul, "{[1]}"']
\end{tikzcd}$$

Here $\pi : \mathrm{Hilb}_V^m \times V \to \mathrm{Hilb}_V^m$ is the projection. By the excess intersection formula [2, Proposition 1.16], we have

$$[[\mathrm{Hilb}_V^m]] = c_{\mathrm{top}}(R^1 \pi_* \mathcal{O}_{\mathbb{C}}(\mathbb{D})) \cap \iota_*[[\mathrm{Hilb}_V^{m-c}]].$$

The complex $R^\bullet\pi_*\mathcal{O}_{\mathbb{C}}(\mathbb{D}))$ is the mapping cone of the morphism

$$R^\bullet\pi_*\mathcal{O}(\mathbb{D}-\mathbb{C}) \to R^\bullet\pi_*\mathcal{O}(\mathbb{D}).$$

Fix a normalized Poincaré line bundle $\mathbb{L}$ on $\mathrm{Pic}^m_V \times V$. Using [2, Lemma 3.15], we see that this choice endows Hilb^m_V with a relatively ample sheaf $\mathcal{O}_{\mathbb{L}}(1)$. Furthermore, there exists an isomorphism

$$\mathcal{O}(\mathbb{D}) \xrightarrow{\cong} (\rho \times \mathrm{id}_V)^*\mathbb{L} \otimes \pi^*\mathcal{O}_{\mathbb{L}}(1),$$

and, since $\mathbb{L}$ is normalized, we have $u = c_1(\mathcal{O}_{\mathbb{L}}(1))$. This implies that $R^\bullet\pi_*\mathcal{O}_{\mathbb{C}}(\mathbb{D})$ is the mapping cone of

$$\rho^*(R^\bullet\mu_*(\mathbb{L}\otimes pr_V^*\mathcal{O}_V(-C)))\otimes\mathcal{O}_{\mathbb{L}}(1) \to \rho^*(R^\bullet\mu_*\mathbb{L})\otimes\mathcal{O}_{\mathbb{L}}(1).$$

Using Corollary 4 we conclude

$$c_{\mathrm{top}}(R^1\pi_*\mathcal{O}_{\mathbb{C}}(\mathbb{D})) = \sum_i \rho^*\left(\frac{\kappa_c^i}{i!}\right)\cdot u^{\frac{1}{2}(c^2+c\cdot m)-m\cdot c-i},$$

which proves part i).

Suppose now that $(k-m)\cdot c < 0$. Then we have $H^1(\mathcal{O}_C(D)) = 0$ for any divisor $D \in \mathrm{Hilb}^{m+c}_V$. Denote by $\tilde{\mathbb{D}} \subset \mathrm{Hilb}^{m+c}_V \times V$ the universal divisor, and let $\tilde{\pi} : \mathrm{Hilb}^{m+c}_V \times V \to \mathrm{Hilb}^{m+c}_V$ be the projection. It follows that the sheaf $R^1\tilde{\pi}_*\mathcal{O}_{\mathrm{Hilb}^{m+c}_V\times C}(\tilde{\mathbb{D}})$ vanishes, and that $\tilde{\pi}_*\mathcal{O}_{\mathrm{Hilb}^{m+c}_V\times C}(\tilde{\mathbb{D}})$ is locally free. Moreover, ι induces an isomorphism

$$\mathrm{Hilb}^m_V \xrightarrow{\cong} Z(\lambda),$$

where λ is the canonical section in $\tilde{\pi}_*\mathcal{O}_{\mathrm{Hilb}^{m+c}_V\times C}(\tilde{\mathbb{D}})$.

The short exact sequence

$$0 \to \mathcal{O}_{\mathbb{D}}(\mathbb{D}) \to \mathcal{O}_{\mathbb{D}+\mathbb{C}}(\mathbb{D}+\mathbb{C}) \to \mathcal{O}_{\mathbb{C}}(\mathbb{D}+\mathbb{C}) \to 0$$

gives rise to the following distinguished triangle:

$$\begin{tikzcd} R^\bullet\pi_*\mathcal{O}_{\mathbb{D}}(\mathbb{D}) \arrow[r] & R^\bullet\pi_*\mathcal{O}_{\mathbb{D}+\mathbb{C}}(\mathbb{D}+\mathbb{C}) \arrow[d] \\ & R^\bullet\pi_*\mathcal{O}_{\mathbb{C}}(\mathbb{D}+\mathbb{C}) \arrow[ul, "{[1]}"] \end{tikzcd}$$

Hence functoriality [4, Theorem 1] yields

$$\iota_*[[\mathrm{Hilb}^m_V]] = c_{\mathrm{top}}(\tilde{\pi}_*\mathcal{O}_{\mathrm{Hilb}^{m+c}_V\times C}(\tilde{\mathbb{D}})) \cap [[\mathrm{Hilb}^{m+c}_V]].$$

Fix again a normalized Poincaré line bundle $\mathbb{L}$ on Pic^m_V. By arguments similar to those of the first part, we see that $R^\bullet\tilde{\pi}_*\mathcal{O}_{\mathrm{Hilb}^{m+c}_V\times C}(\tilde{\mathbb{D}})$ is the mapping cone of

$$\tilde{\rho}^*(R^\bullet\mu_*\mathbb{L})\otimes\mathcal{O}_{\mathbb{L}\otimes pr_V^*\mathcal{O}_V(C)}(1) \to \tilde{\rho}^*(R^\bullet\mu_*(\mathbb{L}\otimes pr_V^*\mathcal{O}_V(C))\otimes\mathcal{O}_{\mathbb{L}\otimes pr_V^*\mathcal{O}_V(C)}(1).$$

Now Corollary 4 implies

$$c_{\mathrm{top}}(\tilde{\pi}_*\mathcal{O}_{\mathrm{Hilb}_V^{m+c}\times C}(\tilde{\mathbb{D}})) = \sum_i \tilde{\rho}^*\left(\frac{(-\kappa_c)^i}{i!}\right)\cdot u^{\frac{1}{2}(c^2+c\cdot k)-(k-m)c-i}. \qquad \square$$

Remark 6. When C is rational, i.e. when the normalization $\hat{C}$ is isomorphic to $\mathbb{P}^1$, then $\kappa_c = 0$. When C is isomorphic to $\mathbb{P}^1$ and $c^2 \in \{0,-1\}$, then $m\cdot c < 0$ or $(k-m)\cdot c < 0$ for any $m \in H^2(V,\mathbb{Z})$.

To see this, let $j : \hat{C} \to V$ be the map induced by the inclusion $C \subset V$. Then for all $a, b \in H^1(V,\mathbb{Z})$

$$\kappa_c(a\wedge b) = \langle a\cup b, j_*[\hat{C}]\rangle = \langle j^*a \cup j^*b, [\hat{C}]\rangle.$$

Since the curve $\hat{C}$ is simply connected, the pull-backs j^*a and j^*b vanish, and therefore $\kappa_c(a\wedge b) = 0$. When C is isomorphic to $\mathbb{P}^1$ and $c^2 \in \{0,-1\}$, the adjunction formula yields $k\cdot c < 0$. This proves the second claim.

3 Relations for Poincaré invariants and the adjunction inequality

First we recall the definition of the Poincaré invariant. Let V be a surface, $p \in V$ an arbitrary point. Fix a class $m \in H^2(V,\mathbb{Z})$, denote by $\mathbb{D}^+$ the universal divisor over the Hilbert scheme Hilb_V^m, and set

$$u^+ := c_1\big(\mathcal{O}(\mathbb{D}^+)|_{\mathrm{Hilb}_V^m\times\{p\}}\big) \in H^2(\mathrm{Hilb}_V^m,\mathbb{Z}).$$

Since V is connected, the class u^+ does not depend on the chosen point p. Likewise, denote by $\mathbb{D}^-$ the universal divisor over the Hilbert scheme Hilb_V^{k-m}, where $k = c_1(\mathcal{K}_V)$. Put

$$u^- := c_1\big(\mathcal{O}(\mathbb{D}^-)|_{\mathrm{Hilb}_V^{k-m}\times\{p\}}\big) \in H^2(\mathrm{Hilb}_V^{k-m},\mathbb{Z}).$$

Denote by $\rho^\pm$ the following morphisms:

$$\begin{aligned}\rho^+ : \mathrm{Hilb}_V^m &\longrightarrow \mathrm{Pic}_V^m \\ D &\longmapsto [\mathcal{O}_V(D)],\end{aligned} \qquad \begin{aligned}\rho^- : \mathrm{Hilb}_V^{k-m} &\longrightarrow \mathrm{Pic}_V^m \\ D' &\longmapsto [\mathcal{K}_V(-D')].\end{aligned}$$

Again we denote the image of $[[\mathrm{Hilb}_V^m]]$ under the cycle map $A_*(\mathrm{Hilb}_V^m) \to H_*(\mathrm{Hilb}_V^m,\mathbb{Z})$ by the same symbol.

Definition 7. Let V be a surface. The *Poincaré invariant* of V is the map

$$\begin{aligned}(P_V^+, P_V^-) : H^2(V,\mathbb{Z}) &\longrightarrow \Lambda^*H^1(V,\mathbb{Z}) \times \Lambda^*H^1(V,\mathbb{Z}) \\ m &\longmapsto (P_V^+(m), P_V^-(m)),\end{aligned}$$

defined by

$$P_V^+(m) := \rho_*^+\left(\sum_i (u^+)^i \cap [[\mathrm{Hilb}_V^m]]\right)$$

and

$$P_V^-(m) := (-1)^{\chi(\mathcal{O}_V)+\frac{1}{2}m(m-k)}\rho_*^-\left(\sum_i (-u^-)^i \cap [[\mathrm{Hilb}_V^{k-m}]]\right),$$

if $m \in \mathrm{NS}(V)$, and by $P_V^\pm(m) := 0$ otherwise.

For an integer n let

$$\tau_n : \Lambda^* H^1(V,\mathbb{Z}) \longrightarrow \Lambda^* H^1(V,\mathbb{Z})$$

be the truncation map introduced above. Recall the identifications

$$H^*(\mathrm{Pic}_V^m,\mathbb{Z}) = \Lambda^* H^1(V,\mathbb{Z})^\vee, \qquad H_*(\mathrm{Pic}_V^m,\mathbb{Z}) = \Lambda^* H^1(V,\mathbb{Z}).$$

Theorem 8. *Let V be a surface, and fix a class $m \in H^2(V,\mathbb{Z})$. Let $C \subset V$ be a reduced and irreducible curve, and set $c := c_1(\mathcal{O}_V(C))$.*

i) *If $m \cdot c < 0$, then*

$$P_V^\pm(m) = \tau_{m(m-k)}\big(\exp(\kappa_c) \cap P_V^\pm(m-c)\big).$$

ii) *If $(k-m)\cdot c < 0$, then*

$$P_V^\pm(m) = \tau_{m(m-k)}\big(\exp(-\kappa_c) \cap P_V^\pm(m+c)\big).$$

Proof. Suppose that $m \cdot c < 0$, and let ι^+ be the inclusion $\mathrm{Hilb}_V^{m-c} \to \mathrm{Hilb}_V^m$. By part i) of Theorem 5 we have

$$\begin{aligned}
P_V^+(m) &= \rho_*^+\left(\sum_i u^i \cap [[\mathrm{Hilb}_V^m]]\right)\\
&= \rho_*^+\left(\sum_i u^i \cap \left(\sum_j (\rho^+)^*\left(\frac{\kappa_c^j}{j!}\right) u^{\frac{1}{2}(c^2+c\cdot k)-m\cdot c-j}\right) \cap \iota_*^+[[\mathrm{Hilb}_V^{m-c}]]\right)\\
&= \sum_j \frac{\kappa_c^j}{j!} \cap \rho_*^+\left(\sum_i u^{i+\frac{1}{2}(c^2+c\cdot m)-m\cdot c-j} \cap \iota_*^+[[\mathrm{Hilb}_V^{m-c}]]\right)\\
&= \tau_{m(m-k)}\big(\exp(\kappa_c) \cap P_V^+(m-c)\big).
\end{aligned}$$

Let ι^- be the inclusion $\mathrm{Hilb}_V^{k-m} \to \mathrm{Hilb}_V^{k-m+c}$, and set $\varepsilon := (-1)^{\chi(\mathcal{O}_V)+\frac{1}{2}m(m-k)}$. Note that under the isomorphism

$$\begin{aligned}
\mathrm{Pic}_V^m &\longrightarrow \mathrm{Pic}_V^{k-m}\\
[\mathcal{L}] &\longmapsto [\mathcal{K}_V \otimes \mathcal{L}^\vee]
\end{aligned}$$

the cohomology class κ_c is mapped to κ_c, since this class is of degree 2. Hence part ii) of Theorem 5 yields

$$\begin{aligned}
&P_V^-(m) = \\
&= \varepsilon \cdot (\rho^-)_* \left(\sum_i (-u)^i \cap \iota_*^- [[\mathrm{Hilb}_V^{k-m}]] \right) \\
&= \varepsilon \cdot \rho_*^- \left(\sum_i (-u)^i \cap \left(\sum_j (\rho^-)^* \left(\frac{(-\kappa_c)^j}{j!} \right) \cdot u^{\frac{1}{2}(c^2+c\cdot k)-m\cdot c-j} \cap [[\mathrm{Hilb}_V^{k-m+c}]] \right) \right) \\
&= \varepsilon \cdot (-1)^{\frac{c^2+c\cdot k}{2}-m\cdot c} \left(\sum_j \frac{\kappa_c^j}{j!} \cap \rho_*^- \left(\sum_i (-u)^{i+\frac{1}{2}(c^2+c\cdot k)-m\cdot c-j} \cap [[\mathrm{Hilb}_V^{k-m+c}]] \right) \right) \\
&= \tau_{m(m-k)}(\exp(\kappa_c) \cap P_V^-(m-c)).
\end{aligned}$$

The proof in the case $(k-m)\cdot c < 0$ is similar. We omit the details. □

Recall that a class $m \in H^2(V,\mathbb{Z})$ is basic for a surface V, if

$$(P_V^+(m), P_V^-(m)) \neq (0,0).$$

The surface V is of simple type if all basic classes $m \in H^2(V,\mathbb{Z})$ satisfy $m(m-k) = 0$. In [2, Proposition 6.25] we have shown that surfaces with $p_g(V) > 0$ are of simple type. The following result can be considered as an algebraic analog of the Ozsváth–Szabó inequality [6, Corollary 1.7].

Proposition 9. *Let V be a surface with $p_g(V) > 0$, let $C \subset V$ be a curve, and set $c := c_1(\mathcal{O}_V(C))$. For any basic class $m \in H^2(V,\mathbb{Z})$ we have*

$$0 \leq m \cdot c \leq k \cdot c,$$

unless C is a smooth rational curve. In this case we have

$$-1 \leq m \cdot c \leq k \cdot c + 1$$

for all basic classes $m \in H^2(V,\mathbb{Z})$.

Proof. Assume first that m is a basic class with $m \cdot c < 0$. Then Theorem 8 implies that also $m - c$ is a basic class. We have

$$\frac{1}{2}(m-c)(m-c-k) = \frac{1}{2}m(m-k) + p_a(C) - 1 - m \cdot c.$$

Since any surface V with $p_g(V) > 0$ is of simple type, this implies

$$p_a(C) = 0 \quad \text{and} \quad m \cdot c = -1.$$

Analogously, if m is a basic class with $m \cdot c > k \cdot c$, then also $m + c$ is a basic class. Because

$$\frac{1}{2}(m+c)(m+c-k) = \frac{1}{2}m(m-k) + p_a(C) - 1 - (k-m) \cdot c,$$

we obtain this time

$$p_a(C) = 0 \quad \text{and} \quad (k-m) \cdot c = -1.$$ □

4 Connection with the Ozsváth–Szabó relation

In order to explain the connection between Theorem 8 and the Ozsváth–Szabó relation, we briefly recall the structure of the full Seiberg–Witten invariants; for the construction and details, we refer to [5].

Let (M, g) be a closed oriented Riemannian 4-manifold with first Betti number b_1. We denote by b_+ the dimension of a maximal subspace of $H^2(M, \mathbb{R})$ on which the intersection form is positive definite. Recall that the set of isomorphism classes of $\mathrm{Spin}^c(4)$-structures on (M, g) has the structure of a $H^2(M, \mathbb{Z})$-torsor. This torsor does, up to a canonical isomorphism, not depend on the choice of the metric g and will be denoted by $\mathrm{Spin}^c(M)$.

We have the Chern class mapping

$$\begin{aligned} c_1 : \mathrm{Spin}^c(M) &\longrightarrow H^2(M, \mathbb{Z}) \\ \mathfrak{c} &\longmapsto c_1(\mathfrak{c}), \end{aligned}$$

whose image consists of all characteristic elements. If $b_+ > 1$, then the Seiberg–Witten invariants are maps

$$SW_{M,\mathcal{O}} : \mathrm{Spin}^c(M) \longrightarrow \Lambda^* H^1(M, \mathbb{Z}),$$

where $\mathcal{O}$ is an orientation parameter. When $b_+ = 1$, then the invariants depend on a chamber structure and are maps

$$(SW^+_{M,(\mathcal{O}_1,\mathbf{H}_0)}, SW^-_{M,(\mathcal{O}_1,\mathbf{H}_0)}) : \mathrm{Spin}^c(M) \longrightarrow \Lambda^* H^1(M, \mathbb{Z}) \times \Lambda^* H^1(M, \mathbb{Z}),$$

where $(\mathcal{O}_1, \mathbf{H}_0)$ are again orientation data. The difference of the two components is a purely topological invariant.

Let $\Sigma \subset M$ be a smoothly embedded, oriented, closed two-manifold. Fix a standard symplectic basis for $H_1(\Sigma, \mathbb{Z})$ and let $\{A_i, B_i\}_{i=1}^g$ be its image in $H^1(M, \mathbb{Z})^\vee$. We define the class $\theta(\Sigma) \in \Lambda^2 H^1(M, \mathbb{Z})^\vee$ by

$$\theta(\Sigma) = \sum_i A_i \wedge B_i.$$

Theorem 10 (Ozsváth–Szabó). *Let M be a closed, oriented, smooth four-manifold with $b_+ > 0$, and let $\Sigma \subset M$ be a smoothly embedded, oriented, closed two-manifold of genus $g > 0$ with negative self-intersection*

$$[\Sigma] \cdot [\Sigma] = -n.$$

If $b_+ > 1$, then for each $\mathrm{Spin}^c(4)$-structure $\mathfrak{c}$ with expected dimension $d(\mathfrak{c}) \geq 0$ and

$$|\langle c_1(\mathfrak{c}), [\Sigma] \rangle| \geq 2g + n$$

we have

$$\mathrm{SW}_{M,\mathcal{O}}(\mathfrak{c}) = \tau_{d(\mathfrak{c})}(\exp(\theta(\varepsilon\Sigma)) \cap \mathrm{SW}_{M,\mathcal{O}}(\mathfrak{c} + \varepsilon\, \mathrm{PD}(\Sigma))),$$

where $\varepsilon = \pm 1$ is the sign of $\langle c_1(\mathfrak{c}), [\Sigma] \rangle$, and $\mathrm{PD}(\Sigma)$ denotes the class Poincaré dual to $[\Sigma]$.

If $b_+ = 1$*, then for each* $\mathrm{Spin}^c(4)$*-structure* $\mathfrak{c}$ *with expected dimension* $d(\mathfrak{c}) \geq 0$ *and*

$$|\langle c_1(\mathfrak{c}), [\Sigma]\rangle| \geq 2g + n$$

we have

$$\mathrm{SW}^{\pm}_{X,(\mathcal{O}_1,\mathbf{H}_0)}(\mathfrak{c}) = \tau_{d(\mathfrak{c})}(\exp(\theta(\varepsilon\Sigma)) \cap \mathrm{SW}^{\pm}_{X,(\mathcal{O}_1,\mathbf{H}_0)}(\mathfrak{c} + \varepsilon\,\mathrm{PD}[\Sigma])).$$

We need the following

Lemma 11. *Let* M *be a closed, oriented, smooth four-manifold. Let* $\Sigma \subset M$ *be a smoothly embedded, oriented, closed two-manifold, and let* c *be the Poincaré dual of the homology class* $[\Sigma]$*. Then*

$$\theta(\Sigma)(a \wedge b) = \langle a \cup b \cup c, [M]\rangle \quad \textit{for all } a, b \in H^1(M, \mathbb{Z}).$$

Proof. Fix a standard symplectic basis $\{\alpha_i, \beta_i\}_{i=1}^g$, and let $\{A_i, B_i\}_{i=1}^g$ be its image in $H^1(V, \mathbb{Z})^\vee$. Then for all $a, b \in H^1(M, \mathbb{Z})$

$$\begin{aligned}\langle a \cup b \cup c, [M]\rangle = \langle a \cup b, c \cap [M]\rangle &= \langle a \cup b, j_*[\Sigma]\rangle = \langle j^*a \cup j^*b, [\Sigma]\rangle \\ &= \sum_{i=1}^{g} \det\begin{pmatrix} j^*a(\alpha_i) & j^*a(\beta_i) \\ j^*b(\alpha_i) & j^*b(\beta_i)\end{pmatrix} \\ &= \sum_{i=1}^{g} \det\begin{pmatrix} A_i(a) & B_i(a) \\ A_i(b) & B_i(b)\end{pmatrix} = \theta(\Sigma)(a \wedge b). \qquad \square\end{aligned}$$

At this point it is clear, that Theorem 8 and Theorem 10 are formally analogous statements. We believe, however, that the actual source of this analogy is the conjectured equivalence between our Poincaré invariants and the full Seiberg–Witten invariants. To be precise, let V be a surface. Any *Hermitian* metric g on V defines a *canonical* $\mathrm{Spin}^c(4)$-structure on (V, g). Its class $\mathfrak{c}_{\text{can}} \in \mathrm{Spin}^c(V)$ does not depend on the choice of the metric. The Chern class of $\mathfrak{c}_{\text{can}}$ is $c_1(\mathfrak{c}_{\text{can}}) = -c_1(\mathcal{K}_V) = -k$.

Since $\mathrm{Spin}^c(V)$ is a $H^2(V, \mathbb{Z})$-torsor, the distinguished element $\mathfrak{c}_{\text{can}}$ defines a bijection:

$$\begin{aligned} H^2(V, \mathbb{Z}) &\longrightarrow \mathrm{Spin}^c(V) \\ m &\longmapsto \mathfrak{c}_m. \end{aligned}$$

The Chern class of the twisted structure $\mathfrak{c}_m$ is $2m - k$. Recall that any surface defines canonical orientation data $\mathcal{O}$ and $(\mathcal{O}_1, \mathbf{H}_0)$ respectively.

The precise conjectured relation between Poincaré and Seiberg–Witten invariants is:

Conjecture 12. *Let* V *be a surface, and denote by* $\mathcal{O}$ *or* $(\mathcal{O}_1, \mathbf{H}_0)$ *the canonical orientation data. If* $p_g(V) = 0$*, then*

$$P^{\pm}_V(m) = \mathrm{SW}^{\pm}_{V,(\mathcal{O}_1,\mathbf{H}_0)}(\mathfrak{c}_m) \quad \textit{for } m \in H^2(V, \mathbb{Z}).$$

If $p_g(V) > 0$*, then*

$$P^{+}_V(m) = P^{-}_V(m) = \mathrm{SW}_{V,\mathcal{O}}(\mathfrak{c}_m) \quad \textit{for } m \in H^2(V, \mathbb{Z}).$$

If this conjecture holds, Theorem 8 is essentially a consequence of Theorem 10. To see this, let $C \subset V$ be an integral curve in the surface V. Its arithmetic genus is given by the adjunction formula

$$p_a(C) = \frac{1}{2}(c^2 + c \cdot k) + 1,$$

where $c := c_1(\mathcal{O}_V(C))$. Hence the inequality

$$|\langle c_1(\mathfrak{c}), [\Sigma]\rangle| \geq 2g + n$$

with $n = -[\Sigma] \cdot [\Sigma]$ reads

$$|\langle c_1(\mathfrak{c}), [\Sigma]\rangle| \geq c \cdot k + 2.$$

When $\mathfrak{c} = \mathfrak{c}_m$ for some $m \in H^2(V, \mathbb{Z})$, this means

$$|(2m - k) \cdot c| \geq c \cdot k + 2,$$

or equivalently

$$m \cdot c \leq -1 \quad \text{or} \quad (k - m) \cdot c \leq -1.$$

Moreover, in the first case $\varepsilon = -1$, whereas in the second case $\varepsilon = +1$.

Conversely, Theorem 8 yields further evidence for the truth of Conjecture 12.

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